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# A combinatorial approximation algorithm for CDMA downlink rate allocation

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# A combinatorial approximation algorithm for CDMA downlink rate allocation*

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**Abstract**

This paper presents a combinatorial algorithm for downlink rate allocation in Code Division Multiple Access (CDMA) mobile networks. By discretizing the coverage area into small segments, the transmit power requirements are characterized via a matrix representation that separates user and system characteristics. We obtain a closed-form analytical expression for the so-called Perron-Frobenius eigenvalue of that matrix, which provides a quick assessment of the feasibility of the power assignment for a given downlink rate allocation. Based on the Perron-Frobenius eigenvalue, we reduce the downlink rate allocation problem to a set of multiple-choice knapsack problems. The solution of these problems provides an approximation of the optimal downlink rate allocation and cell borders for which the system throughput, expressed in terms of utility functions of the users, is maximized.

**Keywords:** CDMA, feasibility transmit power, downlink rate allocation, multiple-choice knapsack, approximation scheme.

**AMS Subject Classifications:** 90B18, 90C27, 90C59

## 1 Introduction

One of the most important features of future wireless communication systems is their support of different user data rates. As a major complicating factor, due to their scarcity, the radio resources have to be used very efficiently. In Code Division Multiple Access (CDMA) systems, transmissions of different terminals are separated using (pseudo) orthogonal codes. The impact of multiple simultaneous calls is an increase in the interference level, that limits the capacity of the system. The assignment of transmission powers to calls is an important problem for network operation, since the interference caused by a call is directly related to the power. In the CDMA downlink, the transmission power is related to the downlink rates. Hence, for an efficient system utilization, it is necessary to adopt a rate allocation scheme in the transmission powers assignment.

The downlink rate assignment problem has been extensively studied in the literature [3, 6, 12, 15, 17]. [6] presents a procedure for finding the power and rate allocations that minimize the total transmit power in one cell. In [12] several rate assignments are analyzed in the context of the trade-off between fairness and over-all throughput. The rates are supposed to be continuos and the algorithms proposed for the rate allocations are based on solving the Lagrangean dual. Another approach for joint optimal rates and powers allocation, based on Perron-Frobenius theory, is presented in [3] and [15]. [3] proposes a distributed algorithm for assigning base station transmitter (BTSs) powers such that the

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common rate of the users is maximized, while in [15] multiple rates are considered. Again, both algorithms assume continuous rates. In [7], a model for characterizing downlink and uplink power assignment feasibility, for a single data rate is presented.

In this paper we propose a rate and power allocation scheme for obtaining a close to optimum throughput for the downlink in a UMTS system located on a highway. In accordance with the UMTS standard, the rates are chosen from a discrete set. Our goal is to assign rates to users, such that the utility of the system is maximised. The utility functions describing the satisfaction of the users have a very general form and do not have to satisfy any convexity requirement. For modeling the network, we use the model proposed in [7], which enables a characterisation of downlink power feasibility via the Perron-Frobenius (PF) eigenvalue of a suitably chosen matrix. Moreover, an explicit analytical expression for the Perron Frobenius eigenvalue can be obtained. This explicit analytical expression of the PF eigenvalue reduces the rate optimization problem to a series of multiple choice knapsack problems, that can be solved efficiently by standard combinatorial optimization techniques. The algorithm we design is actually a fully polynomial time approximation scheme (FPTAS) for the rate optimization problem. The main advantages of this approach are that by considering discrete rates, we avoid the rounding errors due to continuity assumptions and that, given an error bound $\epsilon$, we can find a solution of value at least $(1-\epsilon)$ times the optimum in polynomial time. Moreover, the algorithm can be applied for a very large family of utility functions. Furthermore, our results indicate that the optimization problems for different cells are loosely coupled by a single interference parameter. If this parameter were known, the optimization problems for each cell can be independently solved.

The remainder of this paper is organized as follows. In Section 2 we present the model. In Section 3 we charaterise the existence of a downlink power allocation for a given rate allocation via the Perron Frobenius theory. In Section 4 we formulate the rate optimization problem and present a FPTAS for finding a near optimal solution. We conclude our work and present ideas for further research in Section 5.

## 2 Model

This paper focuses on the modeling of downlink rate allocation in a CDMA system consisting of Base Transmitter Stations (BTSs) along a highway. Specifically, we focus on a two cells model, where only the area between the two basestations is taken into account.

For modelling a cell, we consider the discretized cell model proposed in [7]. Let X and Y be the two basestations, situated at distance $D$ from each other on a highway. The highway is divided into $L$ small segments , from which segments $\{1, ..., I\}$ are assigned to BTS X and segments $\{I+1, ..., L\}$ to BTS Y. We assume that in each segment, the subscribers are located in the middle of the segment and that they have the same data rate and power. Denote by $n_i$ the number of users in segment $i, i \in \{1, ..., L\}$. Without loss of generality, we assume that there is at least one user in each segment.

We model the path loss propagation between a transmitter X and a receiver in segment $i$ by a deterministic path loss propagation model of the following form

$$P_i^{rec} = P_i l_{i,X},$$

where $l_{i,X}$ depends only on the distance $d_i$ between the middle of segment $i$ and BTS X, $P_i^{rec}$ is the received power in the $i$-th segment and $P_i$ is the transmission power towards the $i-th$ segment. If $l_{i,X} = d_i^{-\gamma}$, where $\gamma \geq 0$ is independent on the distance, we obtain the Okumura-Hata model, which performs reasonably in flat service areas (see [1, 10]).

A common measure of the quality of the transmission, is *the energy per bit to interference ratio*, $\left(\frac{E_b}{I_0}\right)$, that, for a user $i$, say, is defined as (see. e.g. [11])

$$\left(\frac{E_b}{I_0}\right)_i = \frac{W}{R_i}\frac{\text{useful signal power received by user i}}{\text{interference +thermal noise}},$$

where $W$ is the system chip rate and $R_i$ is the data rate in segment $i$. Under the described path loss model, with users in the same segment having the same power and the same rate and a constant noise $N_0$, the energy per bit to interference ratio in the segments assigned to BTS X, respectively to BTS Y, become

$$\left(\frac{E_b}{I_0}\right)_i = \frac{W}{R_i}\frac{P_i l_{i,X}}{\alpha l_{i,X}(\sum_{j=1}^{I} n_j P_j - P_i) + l_{i,Y}\sum_{j=I+1}^{L} n_j P_j + N_0}, \text{ for } i \in \{1, ..., I\}, \tag{1}$$

respectively

$$\left(\frac{E_b}{I_0}\right)_i = \frac{W}{R_i}\frac{P_i l_{i,Y}}{\alpha l_{i,Y}(\sum_{j=I+1}^{L} n_j P_j - P_i) + l_{i,X}\sum_{j=1}^{I} n_j P_j + N_0}, \text{ for } i \in \{I+1, ..., L\}. \tag{2}$$

In order to ensure a certain quality of service, the energy per bit to interference ratio in each segment $i$ has to be above a prespecified value $\epsilon_D^*$. In the presence of perfect power control, we can actually assume that in each segment $i$, $\left(\frac{E_b}{I_0}\right)_i = \epsilon_D^*$ .

We measure the satisfaction of a user $i, i \in \{1, ..., L\}$ by means of a positive utility function $u_i(R_i)$. For a presentation of the utility functions commonly used in the literature see [18].

Our goal is to allocate rates from a discrete and finite set R=$\{R_1, ..., R_K\}$ to the users such that the total utility, i.e., the sum of the utilities of all users, is maximized under the condition that the prescribed quality of service is met for all users and that a feasible power assignment exists.

# 3 Downlink transmit power feasibility

In this section we derive a condition for the existence of a feasible power allocation when the rates allocated to users are known. For this, we will make use of the Perron Frobenius theory, by analogy with the characterisation of power feasibility for the uplink in [2, 8, 9].

For a rate allocation $r$, we say that a feasible power assignment exists if there exists a vector $p \in R^L$ verifying the following system

$$\begin{cases} \left(\frac{E_b}{I_0}\right)_i (r, p) = \epsilon_i, \text{ for each user in segment} i \\ p_i \geq 0 \text{ for each } i \in \{1, ..., L\} \end{cases} \tag{3}$$

Before characterising the feasibility of system (3) we introduce some notations. Let $\mathbf{N} = (n_1, ..., n_L)$, $V(r_i) = \frac{\epsilon_D^* r_i}{W + \alpha\epsilon_D^* r_i}$, $\mathbf{L_X} = (l_1, ..., l_I)$ and $\mathbf{L_Y} = (l_{I+1}, ..., l_L)$ , where

$$l_i = \begin{cases} \frac{l_{i,Y}}{l_{i,X}}, & \text{for } i \in \{1, ..., I\} \\ \frac{l_{i,X}}{l_{i,Y}}, & \text{for } i \in \{I+1, ..., L\} \end{cases}$$

Based on (1) and (2), system (3) can be rewritten as:

$$\begin{cases} p_i = \alpha V(r_i) \sum_{j=1}^{I} p_j n_j + V(r_i) l_i \sum_{j=I+1}^{L} p_j n_j + V(r_i) l_{i,X}^{-1} N_0, \text{ for } i \in 1, ..., I \\ p_i = V(r_i) l_i \sum_{j=1}^{I} p_j n_j + \alpha V(r_i) \sum_{j=I+1}^{L} p_j n_j + V(r_i) l_{i,X}^{-1} N_0, \text{ for } i \in I+1, ..., L, \\ p \geq 0 \end{cases} \tag{4}$$

Note that system (4) has $L$ equations, besides the positivity constraint of the power vector. Next we show that the feasibility of (4) is equivalent to the feasibility of a system with 2 equations (each of them characterising one cell) and a positivity constraint.

**Lemma 1** *System (4) is feasible if and only if the following system is feasible:*

$$\begin{cases} \left(1 - \alpha \sum_{j=1}^{I} V(r_i) n_i\right) x - \sum_{j=1}^{I} V(r_j) n_j l_j y = \sum_{j+1}^{I} V(r_j) n_j l_{j,X}^{-1} N_0 \\ - \sum_{j=I+1}^{L} V(r_j) n_j l_j x + \left(1 - \sum_{j=I+1}^{L} V(r_j) n_j\right) y = \sum_{j=I+1}^{L} V(r_j) n_j l_{j,Y}^{-1} N_0 \\ x \geq 0, y \geq 0 \end{cases} \tag{5}$$

**Proof** Let $p$ be a positive solution of (4). In system (4) multiply each equation with the number of users in the corresponding segment and then add the first $I$ equations and then the other $L - I$. It follows that $(x, y) = (\sum_{i=1}^{I} n_i p_i, \sum_{i=I+1}^{L} n_i p_i)$ verifies (5).
Let $(x, y)$ be a solution of (5). Define:

$$p_i = \begin{cases} V(r_i) l_i y + \alpha V(r_i) x + V(r_i) l_{i,X}^{-1} N_0, \text{ for } i \in \{1, ..., I\} \\ V(r_i) l_i x + \alpha V(r_i) y + V(r_i) l_{i,Y}^{-1} N_0 \text{ for } i \in \{I+1, ..., L\} \end{cases} \tag{6}$$

By simple substitution in (4) it can be shown that $p$ is a solution of (4). ■

Lemma 1 reduces the amount of calculations involved in characterising the power feasibility, since it is straightforward to verify that a system with 2 equations in 2 positive variables is feasible.

System (5) can be rewritten in the following form:

$$(\mathbf{I} - \mathbf{T}) \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{c} \tag{7}$$

where

$$\mathbf{T} = \begin{pmatrix} \alpha \sum_{i=1}^{I} V(r_i) n_i & \sum_{i=1}^{I} V(r_i)\ n_i l_i \\ \sum_{i=I+1}^{L} V(r_i) n_i l_i & \alpha \sum_{i=I+1}^{L} V(r_i) n_i \end{pmatrix},$$

$$\mathbf{c} = \begin{pmatrix} \sum\limits_{i=1}^{I} V(r_i) N_0 n_i l_{i,X}^{-1}, \\ \sum\limits_{i=I+1}^{L} V(r_i) N_0 n_i l_{i,Y}^{-1} \end{pmatrix}.$$

Since matrix $\mathbf{T}$ is a non-negative matrix, according to the Perron-Frobenius theorem (see [16]), the feasibility of (7) is determined by the Perron- Frobenius (PF) eigenvalue $\lambda(\mathbf{T})$ of the matrix $\mathbf{T}$ i.e.,

$$p \geq \mathbf{0} \text{ exist and } p = (\mathbf{I} - \mathbf{T})^{-1}\mathbf{c} \quad \Longleftrightarrow \quad \lambda(\mathbf{T}) < 1. \tag{8}$$

The explicit expression of the PF eigenvalue of $\mathbf{T}$ can be calculated easily

$$\lambda(\mathbf{T}) = \frac{1}{2}\left(\sum_{i=1}^{I} \alpha V(r_i) n_i + \sum_{i=I+1}^{L} \alpha V(r_i) n_i\right) + \frac{1}{2}\sqrt{\alpha^2\left(\sum_{i=1}^{I} V(r_i) n_i - \sum_{i=I+1}^{L} V_i n_i\right)^2 + 4\left(\sum_{i=1}^{I} V(r_i) n_i l_i\right)\left(\sum_{i=I+1}^{L} V_i n_i l_i\right)}.$$

Further note that the condition $\lambda(T) < 1$ is equivalent with the following system:

$$\begin{cases} \sum\limits_{i=1}^{I} \alpha V(r_i) n_i + \sum\limits_{i=I+1}^{L} \alpha V(r_i) n_i \leq 2 \\ (1 - \sum\limits_{i=1}^{I} \alpha V(r_i) n_i)(1 - \sum\limits_{i=I+1}^{L} \alpha V(r_i) n_i) > \left(\sum\limits_{i=1}^{I} V(r_i) n_i l_i\right)\left(\sum\limits_{i=I+1}^{L} V_i n_i l_i\right). \end{cases} \tag{9}$$

Since $\sum\limits_{i=1}^{I} \alpha V(r_i) n_i$ and $\sum\limits_{i=I+1}^{L} \alpha V(r_i) n_i$ cannot be both larger then 1 without violating the first inequality of (9), system (9) is equivalent with

$$\begin{cases} \sum\limits_{i=1}^{I} \alpha V(r_i) n_i < 1 \\ \sum\limits_{i=I+1}^{L} \alpha V(r_i) n_i < 1 \\ (1 - \sum\limits_{i=1}^{I} \alpha V(r_i) n_i)(1 - \sum\limits_{i=I+1}^{L} \alpha V(r_i) n_i) > \left(\sum\limits_{i=1}^{I} V(r_i) n_i l_i\right)\left(\sum\limits_{i=I+1}^{L} V_i n_i l_i\right). \end{cases}$$

Hence, we have proved the following theorem.

**Theorem 2** *For a given rate allocation $r$, a feasible power allocation exists, i.e., system (4) is feasible, if and only if*

$$\begin{cases} \sum\limits_{i=1}^{I} \alpha V(r_i) n_i < 1 \\ \sum\limits_{i=I+1}^{L} \alpha V(r_i) n_i < 1 \\ (1 - \sum\limits_{i=1}^{I} \alpha V(r_i) n_i)(1 - \sum\limits_{i=I+1}^{L} \alpha V(r_i) n_i) > \left(\sum\limits_{i=1}^{I} V(r_i) n_i l_i\right)\left(\sum\limits_{i=I+1}^{L} V_i n_i l_i\right). \end{cases}$$

Theorem 2 provides a clear motivation for discretizing the cells into segments, since it facilitates obtaining an analytical model for characterizing the transmit power feasibility for a certain rate allocation and a certain user distribution. Moreover, we observe that the first two conditions we obtained characterize the two cells separately and the third contains products of factors depending only of one cell. In the next section we will show how these nice properties lead to a fast algorithm for finding a close to optimal rate allocation.

# 4 The rate optimization problem

Let $R = \{R_1, R_2, ..., R_K\}$ be the set of admissible rates, where $R_1 < R_2 < ... < R_K$. The decision of dropping the users of a segment is equivalent with assigning zero rate to the respective segment, case in which $R_1 = 0$.

The problem of allocating rates from the set $R$ to users such that the total utility of the users is maximized, under the condition of ensuring the required Quality of Service and a feasible power assignment, can be formulated as follows:

$$(P) \qquad \begin{array}{ll} \max & \sum_{i=1}^{L} u_i(r_i) \\ \text{s.t.} & \left(\frac{E_b}{I_0}\right)_i (r,p) = \epsilon_D^*, \text{ for each user in segment } i \\ & r_i \in \{R_1, ..., R_K\}, \text{ for each } i \in \{1, ..., L\} \\ & p_i \geq 0 \text{ for each } i \in \{1, ..., L\} \end{array}$$

where $r_i$, respectively $p_i$ represent the rate, respectively the power allocated to segment $i$ and $\epsilon_D^*$ is the threshold for the the energy per bit to interference ratio.

We are interested in designing an algorithm for assigning rates to segments in such a way that a throughput of at least $(1 - \epsilon)$ times the optimum is obtained, in a time polynomial in the size of an instance and $\frac{1}{\epsilon}$. Such an algorithm would be a fully polynomial approximation scheme (FPTAS) for problem (P). We distinguish three main steps in the design of the algorithm:

- First we show that finding an optimal solution of (P) can be reduced to solving a set of optimization problems $\{P_1(t), P_2(t) | t \in [t_{min}, t_{max}]\}$, where $P_1(t)$ characterise the first cell, $P_2(t)$ characterise the second cell and the interval $[t_{min}, t_{max}]$ is an interval depending on the system and the user distribution.
- Then we show that $P_1(t)$, respectively $P_2(t)$ are multiple choice knapsack problems, for which efficient algorithms are known.
- Finally, we will prove that, for finding a solution of value at least $(1 - \epsilon)$ times the optimum, for an $\epsilon > 0$, we only have to solve $P_1(t)$ and $P_2(t)$ for $O(\frac{1}{\epsilon})$ $t$'s in $[t_{min}, t_{max}]$. Since to solve $P_1(t)$, respectively $P_2(t)$ we can apply known FPTAS (see e.g. [4]) for the multiple choice knapsack problem, the algorithm we propose is a FPTAS for $(P)$.

We proceed with the first step of the analysis. Theorem 2 implies that the optimization problem (10) is equivalent with the following problem:

$$(P') \qquad \begin{aligned} & \max \sum_{i=1}^{L} u_i(r_i) \\ & \sum_{i=1}^{I} \alpha V(r_i) n_i < 1 \\ & \sum_{i=I+1}^{L} \alpha V(r_i) n_i < 1 \\ & (1 - \sum_{i=1}^{I} \alpha V(r_i) n_i)(1 - \sum_{i=I+1}^{L} \alpha V(r_i) n_i) > \left( \sum_{i=1}^{I} V(r_i) n_i l_i \right) \left( \sum_{i=I+1}^{L} V_i(r_i) n_i l_i \right) \\ & r_i \in \{R_1, ..., R_K\}, \text{ for each } i \in \{1, ..., L\} \end{aligned}$$

Note that if the rate assignment in one of the cells is known, the problem of assigning rates to the segments of the other cell reduces to a multiple choice knapsack problem. The multiple choice knapsack problem is a NP-hard problem, for which a FPTAS based on dynamical programm ing is proposed in [4]. In a multiple choice knapsack problem the following data are given: the sizes and the profits of a set of objects, which are divided into disjoint classes, and the volume of a knapsack. The goal is to choose the set of objects with maximum profit among the sets of objects that fit into the knapsack and contain one object from each class. If, for example, the rates in the cell assigned to BTS Y were known, then, based on $(P')$, the problem of allocating rates to the segments in the cell assigned to BTS X becomes:

$$\begin{aligned} & \max \sum_{i=1}^{L} u_i(r_i) \\ & \sum_{i=1}^{I} V(r_i) n_i \left(\alpha + l_i \frac{\sum_{i=I+1}^{L} \alpha V(r_i) n_i l_i}{1 - \sum_{i=I+1}^{L} \alpha V(r_i) n_i}\right) < 1 \\ & r_i \in \{R_1, ..., R_K\}, \text{ for each } i \in \{1, ..., I\} \end{aligned}$$

This is a multiple choice knapsack problem with the following data: the objects are the pairs $\{(i, s), i \in \{1, ..., I\}, s \in \{1, ..., K\}\}$, a class consists of the objects corresponding to the same segment, the profit of an object $(i, s)$ is $u_i(R_s)$ and its size is $V(r_i)n_i(\alpha + l_i \frac{\sum_{i=I+1}^{L} \alpha V(r_i) n_i l_i}{1 - \sum_{i=I+1}^{L} \alpha V(r_i) n_i})$. The volume of the knapsack is 1.

Hence, if we knew the rate allocation in one of the cells, we could find a rate allocation for the segments in the other cell by applying an algorithm for the multiple choice knapsack problem. Since this also holds for the case where all the segments in one cell receive zero rate, in the following we may assume that in cell X there is at least one segment which receives non-zero rate.

Under these assumptions, problem $(P')$ can be rewritten as:

$$
(P') \quad
\begin{aligned}
&\max \sum_{i=1}^{L} u_i(r_i) \\
&\sum_{i=1}^{I} \alpha V(r_i) n_i < 1 && (10) \\
&\sum_{i=I+1}^{L} \alpha V(r_i) n_i < 1 \\
&\frac{1-\sum_{i=1}^{I} \alpha V(r_i) n_i}{\sum_{i=1}^{I} V(r_i) n_i l_i} > \frac{\sum_{i=I+1}^{L} V_i(r_i) n_i l_i}{1-\sum_{i=I+1}^{L} \alpha V(r_i) n_i} && (11) \\
&\sum_{i=1}^{I} r_i > 0 && (12) \\
&r_i \in \{R_1, ..., R_K\}, \text{ for each } i \in \{1, ..., L\}
\end{aligned}
$$

Constraint (12) ensures that at least one segment in cell X will receive non zero rate. Remark that the variables and parameters characterising the two cells are well separated in $(P')$. This suggests a decomposition of $(P')$ into a set of problems corresponding to the first cell and one corresponding to the second cell. Denote by

$$
t_{min} = \min_{r \in R^L} \frac{\sum_{i=I+1}^{L} V_i(r_i) n_i l_i}{1-\sum_{i=I+1}^{L} \alpha V(r_i) n_i} \text{ and } t_{max} = \max_{r \in R^L, r \neq 0} \frac{1-\sum_{i=1}^{I} \alpha V(r_i) n_i}{\sum_{i=1}^{I} V(r_i) n_i l_i}
$$

where $R'$ is the smallest nonzero rate.

From (10)-(12) follows that (P') is feasible if and only if $\alpha V(R_1) \min_{i \in \{I+1,...,L\}} n_i l_i < 1$ and $t_{min} \leq t_{max}$. In what follows, we suppose that these two conditions are always satisfied. For each $t \in [t_{min}, t_{max}]$ consider the following problems:

$$
P_1(t) \quad
\begin{aligned}
&\max \sum_{i=1}^{I} u_i(r_i) \\
&\frac{1-\sum_{i=1}^{I} \alpha V(r_i) n_i}{\sum_{i=1}^{I} V(r_i) n_i l_i} > t \\
&\sum_{i=1}^{I} r_i > 0 \\
&r_i \in \{R_1, ..., R_K\}, \text{ for each } i \in \{1, ..., I\}
\end{aligned}
$$

and

$$P_2(t) \qquad \begin{aligned} &\max \sum_{i=1}^{L} u_i(r_i) \\ &t > \frac{\sum\limits_{i=I+1}^{L} V_i(r_i) n_i l_i}{1 - \sum\limits_{i=I+1}^{L} \alpha V(r_i) n_i} \\ &r_i \in \{R_1, ..., R_K\}, \text{ for each } i \in \{I+1, ..., L\} \end{aligned}$$

Let $OPT$ denote the optimal value of the optimization problem $(P')$ and $OPT_1(t)$, respectively $OPT_2(t)$, be the optimal values of $P_1(t)$, respectively $P_2(t)$. In the following Lemma we prove that we can find $OPT$ by solving $P_1(t)$ and $P_2(t)$ for all $t \in [t_{min}, t_{max}]$.

**Lemma 3** $OPT = \max\limits_{t \in [t_{min}, t_{max}]} OPT_1(t) + OPT_2(t)$

**Proof** Consider a $t \in [t_{min}, t_{max}]$. Let $(\bar{r}_1, ..., \bar{r}_I)$, respectively $(\tilde{r}_{I+1}, ..., \tilde{r}_L)$, be optimal solutions of $P_1(t)$, respectively $P_2(t)$. Clearly, $(\bar{r}_1, ..., \bar{r}_I, \tilde{r}_{I+1}, ..., \tilde{r}_L)$ is a feasible solution of $(P')$, and therefore $OPT_1(t) + OPT_2(t) \leq OPT$. We proved that $\max\limits_{t \in [t_{min}, t_{max}]} OPT_1(t) + OPT_2(t) \leq OPT$.

In order to prove the reverse inequality, consider an optimal solution $r^*$ of $(P)$. Let $t = \frac{1 - \alpha \sum_{i=1}^{I} V(r_i^*) n_i}{\sum_{i=1}^{I} V(r_i^*) n_i p_i}$. Since $(r_1^*, ..., r_I^*)$ is feasible for $P_1(t)$ and $(r_{I+1}^*, ..., r_L^*)$ is feasible for $P_2(t)$, $OPT \leq OPT_1(t) + OPT_2(t)$. ■

Lemma 3 implies that an optimal rate allocation can be found by solving independently the set of optimization problems $\{P_1(t) | t \in [t_{min}, t_{max}]\}$ and $\{P_2(t) | t \in [t_{min}, t_{max}]\}$ where each set characterizes only one cell, the cells interracting only through the parameter $t$.

Next we show that $P_1(t)$ and $P_2(t)$ are multiple choice knapsack problems, which can be efficiently solved. For this, we rewrite $P_1(t)$ and $P_2(t)$ in the following form:

$$P_1(t) \qquad \begin{aligned} &\max \sum_{i=1}^{I} u_i(r_i) \\ &\sum_{i=1}^{I} V(r_i) n_i (\alpha + l_i t) < 1 \\ &\sum_{i=1}^{I} r_i > 0 \\ &r_i \in \{R_1, ..., R_K\}, \text{ for each } i \in \{1, ..., I\} \end{aligned}$$

and

$$P_2(t) \qquad \begin{aligned} &\max \sum_{i=1}^{L} u_i(r_i) \\ &\sum_{i=I+1}^{L} V(r_i) n_i(\alpha t + l_i) < t \\ &r_i \in \{R_1, ..., R_K\}, \text{ for each } i \in \{I+1, ..., L\} \end{aligned}$$

The input to the multiple choice knapsack problems $P_1(t)$, respectively $P_2(t)$ is: the objects are the pairs $\{(i,s), i \in \{1,...,I\}, s \in \{1,...,K\}\}$, respectively $\{(i,s), i \in \{I+1,...,L\}, s \in \{1,...,K\}\}$; a class consists of the objects corresponding to the same segment; the profit of an object $(i,s)$ is $u_i(R_s)$ and its size is $V(r_i)n_i(\alpha + l_i t)$ for $i \in \{1,...,I\}$, respectively $V(r_i)n_i(\alpha t + l_i)$ for $i \in \{I+1,...,L\}$. The volumes of the knapsacks are 1, respectively $t$. In $P_1(t)$ an extra condition is imposed, namely that the zero rate cannot be allocated to all users in cell X.

Since $P_1(t)$ and $P_2(t)$ are multiple choice knapsack problems, close to optimal solutions can be found by applying for example the FPTAS described in [4]. For an $\epsilon > 0$ and $t \in [t_{\min}, t_{\max}]$, let $K_1(t,\epsilon)$ and $K_2(t,\epsilon)$, be the value of the solution given by a FPTAS for $P_1(t)$, respectively $P_2(t)$. Hence,

$$K_1(t,\epsilon) \geq (1-\epsilon)OPT_1(t)$$

and

$$K_2(t,\epsilon) \geq (1-\epsilon)OPT_2(t).$$

Let $t^*$ be the value for which $OPT_1(t^*) + OPT_2(t^*) = OPT$.

In next Lemma we will prove that a feasible solution of $(P')$ of value at least $(1-\epsilon)OPT$ can be found using only the values $K_1(t,\epsilon)$ and $K_2(t,\epsilon)$, for $t \in [t_{min}, t_{max}]$.

**Lemma 4** *For each $\epsilon > 0$, the following relation holds*

$$\max_{t \in [t_{min}, t_{max}]} \{K_1(t,\epsilon) + K_2(t,\epsilon)\} \geq (1-\epsilon)OPT.$$

**Proof** From Lemma 3 follows

$$\begin{aligned} \max_{t \in [t_{min}, t_{max}]} \{K_1(t,\epsilon) + K_2(t,\epsilon)\} &\geq K_1(t^*,\epsilon) + K_2(t^*,\epsilon) \\ &\geq (1-\epsilon)OPT_1(t^*) + (1-\epsilon)OPT_2(t^*) \\ &\geq (1-\epsilon)OPT, \end{aligned}$$

where for the second inequality we have used that $K_1(t^*,\epsilon)$, respectively $K_2(t^*,\epsilon)$ are values returned by a FPTAS for $P_1(t^*)$, respectively $P_2(t^*)$. ■

However, if $\epsilon \geq \frac{1}{2}$, in order to find a solution of value $(1-\epsilon)OPT$ it is not necessary to calculate $\max_{t \in [t_{min}, t_{max}]} \{K_1(t,\epsilon) + K_2(t,\epsilon)\}$, as it is shown in the following Lemma:

**Lemma 5** *If $\epsilon \geq \frac{1}{2}$, then $max\{K_1(t_{min}, 2\epsilon - 1) + K_2(t_{min}, 2\epsilon - 1), K_1(t_{max}, 2\epsilon - 1) + K_2(t_{max}, 2\epsilon - 1)\} \geq (1-\epsilon)OPT$.*

**Proof**

First remark that $OPT_1(t)$ is a decreasing function of $t$, while $OPT_2(t)$ is an increasing function of $t$. Hence, $OPT_1(t_{min}) \geq OPT_1(t^*)$ and $OPT_2(t_{max}) \geq OPT_2(t^*)$. For $\epsilon > \frac{1}{2}$ the following holds:

$$max\{K_1(t_{min}, 2\epsilon - 1) + K_2(t_{min}, 2\epsilon - 1), K_1(t_{max}, 2\epsilon - 1) + K_2(t_{max}, 2\epsilon - 1)\} \geq$$
$$\geq \frac{1}{2}\left(K_1(t_{min}, 2\epsilon - 1) + K_2(t_{min}, 2\epsilon - 1) + K_1(t_{max}, 2\epsilon - 1) + K_2(t_{max}, 2\epsilon - 1)\right) \geq$$
$$\geq (1 - \epsilon)\left(OPT_1(t_{min}) + OPT_2(t_{max})\right) \geq (1 - \epsilon)OPT$$

■

In other words, if $\epsilon \geq 1/2$, a feasible solution of $(P')$ of value at least $(1-\epsilon)OPT$ can be found just by calculating $K_1(t_{min}, 2\epsilon - 1)$ and $K_2(t_{max}, 2\epsilon - 1)$.

In the sequel, we suppose that $\epsilon < \frac{1}{2}$.

The only bottleneck in finding a solution of $(P')$ of value at least $(1-\epsilon)OPT$ is that we have to calculate $K_1(t\epsilon)$ and $K_2(t\epsilon)$ for all $t \in [t_{min}, t_{max}]$. However, as we will see below, we can still obtain a solution close to optimum by analysing only a polynomial number of values of $t$.

For $\epsilon > 0$, let $t_{app}$ be the value of $t$ for which

$$K_1(t_{app}, \epsilon) + K_2(t_{app}, \epsilon) = \max_{t \in [t_{min}, t_{max}]} \{K_1(t, \epsilon) + K_2(t, \epsilon)\}.$$

Note that $OPT_1(t)$, respectively $OPT_2(t)$ are step functions and have at most $2^{KI}$, respectively $2^{KJ}$ jump points, the number of the possible rate assignments in each cell. Therefore, for finding $t_{app}$, it would suffice to check only the jump points of the two functions.

Next lemma's further reduce the set of $t$'s that must be considered for obtaining a solution of value at least $(1-\epsilon)OPT$.

**Lemma 6** *For each $\epsilon < \frac{1}{2}$, the following holds*

$$t_{app} \in [t_{min}, t_{max}] \setminus \{t | K_1(t_{app}, \epsilon) < \epsilon K_1(t_{min}, \epsilon) \text{ and } K_2(t_{app}, \epsilon) < \epsilon K_2(t_{max}, \epsilon)\}$$

*either* $K_1(t_{app}, \epsilon) > \epsilon K_1(t_{min}, \epsilon)$ *or* $K_2(t_{app}, \epsilon) > \epsilon K_2(t_{max}, \epsilon)$.

**Proof** Suppose that $K_1(t_{app}, \epsilon) < \epsilon K_1(t_{min}, \epsilon)$ and $K_2(t_{app}, \epsilon) < \epsilon K_2(t_{max}, \epsilon)$. Hence,

$$K_1(t_{app}, \epsilon) + K_2(t_{app}, \epsilon) < \epsilon\left(K_1(t_{min}, \epsilon) + K_2(t_{max}, \epsilon)\right),$$

which, since $\epsilon < \frac{1}{2}$, leads to a contradiction with

$$K_1(t_{app}, \epsilon) + K_2(t_{app}, \epsilon) \geq \frac{1}{2}\left(K_1(t_{min}, \epsilon) + K_2(t_{min}, \epsilon) + K_1(t_{max}, \epsilon) + K_2(t_{max}, \epsilon)\right).$$

■

Consider the sets $A_l(\epsilon)$ and $\overline{A_l}(\epsilon)$, for $l \in \{0, 1, ..., \lfloor \frac{1}{\epsilon} ln \frac{1}{\epsilon} \rfloor + 1\}$ defined as

$$\begin{aligned}
A_0(\epsilon) &= \{t | K_1(t_{min}, \epsilon) < K_1(t, \epsilon)\} \\
\overline{A_0}(\epsilon) &= \{t | K_2(t_{max}, \epsilon) < K_2(t, \epsilon)\} \\
A_l(\epsilon) &= \{t | (1-\epsilon)^l K_1(t_{min}, \epsilon) < K_1(t, \epsilon) < (1-\epsilon)^{l-1} K_1(t_{min}, \epsilon)\}, \text{ for } l \geq 1 \\
\overline{A_l}(\epsilon) &= \{t | (1-\epsilon)^l K_2(t_{max}, \epsilon) < K_2(t, \epsilon) < (1-\epsilon)^{l-1} K_2(t_{max}, \epsilon)\}, \text{ for } l \geq 1
\end{aligned}$$

**Remark 7** *From the fact that* $(1-\epsilon)^{\frac{1}{\epsilon} ln \frac{1}{\epsilon}} < \epsilon$*, and from Lemma 6 follows that* $t_{app} \in \bigcup_{l=0}^{\lfloor \frac{1}{\epsilon} ln \frac{1}{\epsilon} \rfloor + 1} (A_l(\epsilon) \cup \overline{A_l}(\epsilon))$

Further we will prove that by choosing only one element from each set $A_l$, respectively $\overline{A_l}$, we will not deviate significantly from the optimum. This will reduce the number of $t$'s to consider to at most $\lfloor \frac{2}{\epsilon} ln \frac{1}{\epsilon} \rfloor + 2$.

**Lemma 8** *a) If* $t_{app} \in A_l(\epsilon)$*, then for each* $t \in A_l(\epsilon)$*,* $(1-\epsilon) K_1(t_{app}, \epsilon) \leq K_1(t, \epsilon)$*.*
*b) If* $t_{app} \in \overline{A_l}(\epsilon)$*, then for each* $t \in \overline{A_l}(\epsilon)$*,* $(1-\epsilon) K_2(t_{app}, \epsilon) \leq K_2(t, \epsilon)$

**Proof** a) For $l = 0$,

$$K_1(t_{min}, \epsilon) \geq (1-\epsilon) OPT_1(t_{min}) \geq (1-\epsilon) OPT_1(t_{app}) \geq (1-\epsilon) K_1(t_{app}, \epsilon),$$

where for the second inequality we used the monotonicity of $OPT_1$. For $l \in \{1, ..., \lfloor \frac{1}{\epsilon} ln \frac{1}{\epsilon} \rfloor + 1\}$ the proof follows immediately from the definition of $A_l$. ■

Let $J_1(\epsilon)$ be the set containing the maximal element from each nonempty set $A_l(\epsilon)$ and $J_2(\epsilon)$ the set containing the minimal element from each nonempty set $\overline{A_l}(\epsilon)$.

The following lemma shows that in order to find a feasible solution of $(P)$ of value at least $(1-\epsilon)OPT$ it is enough to calculate $K_1(t, \epsilon')$ and $K_2(t, \epsilon')$ only for $t \in J_1(\epsilon') \cup J_2(\epsilon')$, for a well chosen $\epsilon'$.

**Lemma 9** *For* $\epsilon' = 1 - \sqrt[3]{1-\epsilon}$ *the following relation holds*

$$\max_{t \in J_1(\epsilon') \cup J_2(\epsilon')} \{K_1(t, \epsilon') + K_2(t, \epsilon')\} \geq (1-\epsilon) OPT.$$

**Proof** We have seen in Remark 7 that $t_{app} \in \bigcup_{l=0}^{\lfloor \frac{1}{\epsilon'} ln \frac{1}{\epsilon'} \rfloor + 1} (A_l(\epsilon') \cup \overline{A_l}(\epsilon'))$. Suppose that $t_{app} \in A_k(\epsilon') \cap \overline{A_l}(\epsilon')$. Let $t_k = J_1(\epsilon') \cap A_k(\epsilon')$ and $\overline{t_l} = J_2(\epsilon') \cap \overline{A_l}(\epsilon')$.

From Lemma 8 follows that

$$K_1(t_k, \epsilon') \geq (1-\epsilon') K_1(t_{app}, \epsilon') \tag{13}$$

and

$$K_2(\overline{t_l}, \epsilon') \geq (1-\epsilon') K_2(t_{app}, \epsilon'). \tag{14}$$

Suppose that $t_k \geq \overline{t_l}$. Since $OPT_2(t)$ is an increasing function, the following relations hold:

$$\begin{aligned}
K_2(t_k, \epsilon') &\geq (1-\epsilon') OPT_2(t_k) \geq (1-\epsilon') OPT_2(\overline{t_l}) \\
&\geq (1-\epsilon') K_2(\overline{t_l}, \epsilon')
\end{aligned} \tag{15}$$

Combining (13), (14), (15) and Lemma 4, we obtain

$$\begin{aligned} K_1(t_k,\epsilon') + K_2(t_k,\epsilon') &\geq (1-\epsilon')(K_1(t_k,\epsilon') + K_2(\overline{t_l},\epsilon')) \\ &\geq (1-\epsilon')^2(K_1(t_{app},\epsilon') + K_2(t_{app},\epsilon')) \\ &\geq (1-\epsilon')^3 OPT, \end{aligned}$$

where the first inequality follows from (13), the second from (14) and (15), and the third from Lemma 4. Substituting $\epsilon' = 1 - \sqrt[3]{1-\epsilon}$ in the last relation, we get

$$\max_{t \in J_1(\epsilon') \cup J_2(\epsilon')} \{K_1(t,\epsilon') + K_2(t,\epsilon')\} \geq (1-\epsilon) OPT.$$

A similar anaysis can be done if $t_k \leq \overline{t_l}$, but based on the monotonicity of $OPT_1(t)$. ■

Hence, the number of points we are looking at in order to find a solution close to the optimum is reduced to $|J_1(\epsilon)| + |J_2(\epsilon)| = \frac{2}{\epsilon'} ln \frac{1}{\epsilon'} + 2 = O(\frac{1}{\epsilon'} ln \frac{1}{\epsilon'})$. Note that the points in $J_1(\epsilon') \cup J_2(\epsilon')$ can be found in polynomial time by the search procedure described in [13], where at every query, a FPTAS for the multiple choice knapsack problem is performed. This implies that the procedure of finding $\max_{t \in J_1(\epsilon') \cup J_2(\epsilon')} \{K_1(t,\epsilon') + K_2(t,\epsilon')\}$ runs in polynomial time in the size of the instance and in $\frac{1}{\epsilon}$ and that the following procedure is a FPTAS for problem (P):

- Let $\epsilon' = 1 - \sqrt[3]{1-\epsilon}$.
- Find the sets $J_1(\epsilon')$ and $J_2(\epsilon')$.
- For all $t \in J_1(\epsilon') \cup J_2(\epsilon')$, calculate $K_1(t,\epsilon')$ and $K_2(t,\epsilon')$, by using a FPTAS for the multiple choice knapsack problem.
- Choose the $t \in J_1(\epsilon') \cup J_2(\epsilon')$ for which $\max_{t \in J_1(\epsilon') \cup J_2(\epsilon')} \{K_1(t,\epsilon') + K_2(t,\epsilon')\}$ is attained.

## 5 Summary and Further Research

This paper has provided a combinatorial algorithm for finding a downlink rate allocation in a CDMA network, that, for an $\epsilon > 0$, achieves a throughput of value at least $(1-\epsilon)$ times the optimum. Based on the Perron-Frobenius eigenvalue of the power assignment matrix, we have reduced the downlink rate allocation problem to a set of multiple-choice knapsack problems, for which efficient algorithms are known. This approach proves to have several advantages. First, the discrete optimization approach has eliminated the rounding errors due to continuity assumptions of the downlink rates. Using our model, the exact rate that should be allocated to each user can be indicated. Second, the rate allocation approximation we proposed guarantees that the solution obtained is close to the optimum. Moreover, the algorithm works for very general utility functions. Furthermore, our results indicate that the optimal downlink rate allocation can be obtained in a distributed way: the allocation in each cell can be optimised independently, interference being incorporated in a single parameter $t$.

It is among our aims for further research to develop a downlink rate algorithm that takes into account mobility of users and limited transmit powers of cells.

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